Saturation effects in three-level laser systems with constant loss

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(Received 2 July 1981; accepted for publication 3 February 1982)

A quasi-steady-state analysis of a pulsed three-level laser system (homogeneously broadened) was extended to account for uniform losses along the cavity. It was found that in the low pumping rate limit such losses do not affect results previously obtained for systems without losses. In the high pumping rate limit the losses tend to reduce the spatial effects expressed by the correlation function, but the coefficient of the output radiation term, which is usually unity without losses, is decreased. Therefore, the output energy term in the spatially averaged rate equations, in the case where losses are incorporated, should be modified. The analytical results were confirmed by numerical solution of the exact (one-dimensional) space-time equations relevant to this system. Comparisons with other treatments were also carried out and the agreement obtained was satisfactory.

PACS numbers: 42.55.Bi, 42.60.By

I. INTRODUCTION

In previous works\(^1\) three-level laser systems were treated in order to find the conditions for which the rate equations, obtained by applying the quasi-steady-state (QSS) assumption to the space-time equations, are reliable. The space-time rate equations were solved numerically and the results were compared with those obtained from the spatially averaged equations modified in accordance with the QSS assumption. This comparison revealed the following:

(a) the existence of two disconnected quasi-steady-state regions, namely the small signal gain (SSG) region and the high pumping (HP) region, which are well described within the QSS model which separates the SSG and HP regions;

(b) the existence of an intermediate region which is too complicated to be described within the QSS model; and

(c) the spatially independent equations for the HP region are similar in form to those for the SSG region except that one constant, which in the SSG is unity, becomes smaller than one in the HP region.

In the present work the previous study is extended to include constant losses in the cavity. This extension is trivial for the case of the SSG region, but becomes much more involved when the system is assumed to be in the HP region. Similar studies appeared recently\(^3\)-\(^6\) but these deal with cw lasers.

In Sec. II, we present the general (one-dimensional) space-time equations. The QSS model is introduced in Sec. III, while in Sec. IV numerical results are shown and discussed. In Sec. V comparisons with experimental and other theoretical results are presented.

II. THE ONE-DIMENSIONAL SPACE-TIME DEPENDENT EQUATIONS

The one-dimensional space-time dependent equations which govern the behavior of a homogeneously broadened three-level laser system, and which contain a loss term are\(^2\)-\(^11\)

\[
\frac{1}{c} \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x} + (n - g_i)\phi + S, \tag{1}
\]

\[
\frac{\partial n_i}{\partial t} = G_i - \gamma (r^{-1} n_2 + c n \phi), \quad i = 1, 2, \tag{2}
\]

where \(\phi = \sigma p \pm; p \pm\) represents photon densities and \(\sigma\) is the cross section for stimulated emission (the plus sign indicates photons moving to the right and the minus sign indicates those moving to the left); \(S\) represents the spontaneous emission term, \(n_i (i = 1, 2)\) the lower and the upper laser level populations densities, respectively (multiplied by \(\sigma\)); \(n = n_2 - n_1, \phi = \phi^{+} + \phi^{-}; \gamma\) is the collisional deactivation time of the upper laser level, \(c\) the velocity of light; \(G_i (i = 1, 2)\) is equal to \(\sigma P_i (i = 1, 2)\), where \(P_i\) represents the pumping rates of the corresponding levels, and \(g_i\) is the loss coefficient.

Integration of Eqs. (1) and (2) with respect to \(x\) from \(x = 0\) to \(x = L\), and division by \(L\) (the cavity length) leads to

\[
\frac{1}{c} \frac{\partial \phi}{\partial t} = (q(t) \bar{n} - g_i)\phi - Q(t) + \bar{S}, \tag{3}
\]

\[
\frac{\partial n_i}{\partial t} = \bar{G}_i - \gamma (r^{-1} n_2 + q(t) \bar{n} \phi), \quad i = 1, 2, \tag{4}
\]

where the bars indicate average values with respect to \(x\) for any time \(t\), and \(q\) is a correlation function (CF) defined as:

\[
q(t) = \frac{n(t)}{\bar{n} \phi}, \tag{5}
\]

and \(Q(t)\) is the output energy term which can be shown to be

\[
Q(t) = \left[ \zeta (L, t) - \zeta (0, t) \right] \frac{1}{L}. \tag{6}
\]

Here,

\[
\zeta (x, t) = \phi^+ (x, t) - \phi^- (x, t), \tag{7}
\]

which together with \(\phi (x, t)\) form a set of two independent functions.
III. THE QUASI-STeady-STATE MODEL (QSM): THEORY

We now consider the QSSM\(^1\) to\(^2\) in order to obtain \(q(t)\) and \(Q(t)\) for the case where losses cannot be ignored. The relevant equations to be treated are

\[
\begin{align*}
\frac{d\phi}{dx} &= \left(\frac{g_t}{1 + \phi} - g_i\right) \xi, \\
\frac{d\xi}{dx} &= \left(\frac{g_t}{1 + \phi} - g_i\right) \phi.
\end{align*}
\]  (8)

Here, \(g_t\) can be interpreted as the instantaneous small signal gain, \(\phi\) and \(\xi\) are normalized with respect to the instantaneous saturation intensity and, consequently, are dimensionless. Multiplying the first equation by \(\phi\) and the second by \(\xi\), subtracting the second from the first and integrating, we obtain

\[
\phi^2 - \xi^2 = \rho^2,
\]  (9)

where \(\rho\) is a constant that can be determined from the boundary conditions to be

\[
\rho = \frac{2}{1 + R_0} \frac{\phi(0)}{(R_L)^{1/2}} - \frac{(R_L)^{1/2}}{1 + R_L} \phi(L).
\]  (10)

Substituting Eqs. (9) and (10) into Eq. (6), one can express the output energy function \(Q(t)\) in a much simpler form, i.e.,

\[
Q(t) = \frac{1}{L} \frac{\phi(L)}{\lambda_L}, \quad \lambda_L = \frac{(1 + R_L)(R_0)^{1/2}}{(1 - R_0)(R_L)^{1/2} + (1 - R_L)(R_0)^{1/2}}.
\]  (11)

Equation (9) is also used to eliminate \(\xi(x)\) from Eq. (8) to obtain an equation for \(\phi(x)\) only:

\[
\frac{d\phi}{(\phi^2 - \rho^2)^{1/2}} = \left(\frac{g_t}{1 + \phi} - g_i\right) dx.
\]  (12)

We next consider two cases:

(a) \textit{Low pumping rate limit}. Multiplying Eq. (12) once by \((1 + \phi)\), and averaging over \(x\) yields:

\[
g_t - g_i - g_i \tilde{\phi} = \frac{1}{L} \int \frac{1 + \phi}{(\phi^2 - \rho^2)^{1/2}} d\phi,
\]  (13)

and once by \(\phi(1 + \phi)\) and again averaging over \(x\), we obtain

\[
(\phi - g_i) \tilde{\phi} - g_i \phi^2 = \frac{1}{L} \int \frac{\phi(1 + \phi)}{(\phi^2 - \rho^2)^{1/2}} d\phi.
\]  (14)

Since it is assumed that the pumping rate is relatively low, we may neglect \(g, \tilde{\phi}\) in Eq. (13) and \(g_i, \phi^2\) in Eq. (14). Thus, by solving the corresponding integrals, one finds

\[
g_t - g_i = \frac{1}{L} (\beta + \text{higher order terms}),
\]  (15)

where

\[
\beta = -\frac{1}{2} \ln(R_0 R_L).
\]  (16)

Substituting Eq. (15) into Eq. (16), we obtain

\[
\tilde{\phi} = \frac{\phi(L)}{\lambda_L \beta},
\]  (17)

which is the same result as for \(g_t = 0, \beta\)\(^1\)\(^2\).

Calculation of \(\tilde{n}\) and \(n\phi\) as was done in Ref. 1, gives

\[
\tilde{n} = g_t + \frac{1}{L} \beta,
\]  (18)

and

\[
\frac{n \phi}{g_t} = g_t \tilde{\phi} + \frac{1}{L} \frac{\phi(L)}{\lambda_L}.
\]  (19)

Substitution of Eqs. (18), (19), and (20) in Eq. (5) gives \(q(t) = 1\) for the correlation function.

As in the previous work,\(^1\) we also introduce the normalized output function \(\tilde{q}\), defined as

\[
\tilde{q} = \frac{L Q}{\beta \tilde{n}}.
\]  (21)

Substituting Eq. (11) and (18) into Eq. (21), we obtain \(\tilde{q} = 1\).

These results indicate that in the low pumping rate limit, no matter how large the losses along the cavity are, the basic equations are not affected.

(b) \textit{High pumping rate limit}. To treat this case we introduce a new variable \(\alpha\),

\[
\alpha = g\tilde{t} - g_i.
\]  (22)

Then, from Eq. (12), we obtain

\[
g_t dx = \frac{(1 + \phi) d\phi}{(\phi^2 - \rho^2)^{1/2} (\alpha - \phi)}. \quad (23)
\]

Equation (23) is first integrated to give

\[
\frac{1}{g_t L} \int \frac{(1 + \phi) d\phi}{(\phi^2 - \rho^2)^{1/2} (\alpha - \phi)} = 1,
\]  (24)

and then Eq. (23) is multiplied by \(\phi\) and integrated to give

\[
\frac{1}{g_t L} \int \frac{(1 + \phi) d\phi}{(\phi^2 - \rho^2)^{1/2} (\alpha - \phi)} = \bar{\Phi}.
\]  (25)

Defining \(F(\alpha)\) as

\[
F(\alpha) = \int \frac{d\phi}{(\phi^2 - \rho^2)^{1/2} (\alpha - \phi)},
\]  (26)

it can be shown that

\[
(\phi - \alpha) g_t L + \beta + \frac{\phi(L)}{\lambda_L} = 0.
\]  (27)

Solving Eqs. (24) and (25) and applying the definition of \(F(\alpha)\), we obtain

\[
(1 + \alpha) F(\alpha) - (\beta + g_t L) = 0,
\]  (28)
Equations (27) and (28) implicitly contain the relation between $\alpha$ and $\phi (L)$. Once $\alpha (\phi (L))$ is known, Eq. (29) yields the relation between $\phi$ and $\phi (L)$. The explicit relation between $\phi$ and $\phi (L)$ (in the high pumping rate limit) can be determined analytically to a certain extent as follows. Equation (29) indicates that $\phi$ is always smaller than $\alpha$. In fact, considering Eq. (12) (or, for that matter, Eq. (23)) it can be seen that the system cannot reach a (quasi) steady state unless $\phi (x)$ is bounded by $\alpha$. Thus,

$$\max (\phi (x)) \leq \alpha.$$  

Assuming $R_t < R_{o}$, it can be seen that

$$\max (\phi (x)) = \phi (L).$$  

Therefore, we make the ansatz that

$$\alpha = \phi (L).$$  

Substituting $\phi (L)$ for $\alpha$ in Eq. (29), we find

$$\tilde{\phi} = \phi (L) - \frac{1}{g_i L} \left( \frac{\phi (L)}{\lambda_L} + \beta \right).$$  

It can be seen from Eq. (33) that Eq. (32) is, at most, relevant when $g_i$ is large, but breaks down when $g_i \rightarrow 0$. Therefore, our next step is to "correct" Eq. (32) as follows

$$\alpha = \phi (L) + \frac{1}{g_i L} \left( \frac{\phi (L)}{\lambda_L} + \beta \right),$$  

which (hopefully) will extend the validity of this representation. However, since for $g_i \rightarrow 0$, $\phi$ in the high saturation limit is known to be

$$\tilde{\phi} = \phi (L) \frac{\lambda}{\lambda_L};$$

$$\lambda = \frac{(R_0 + R_L)[1 - R_0 R_L] + 4R_0 R_L \beta}{2[(R_0)^{1/2} + (R_L)^{1/2}]^2[1 - (R_0 - R_L)^{1/2}].}$$

$\alpha$ has to be still further modified, as follows

$$\alpha = \phi (L) + \frac{1}{g_i L} \left( \frac{\phi (L)}{\lambda_L} + \beta \right) + \phi (L) \frac{\lambda - \lambda_L}{\lambda_L} H (g_i L),$$

where $H (g_i L)$ is defined as

$$H (g_i L) \rightarrow \begin{cases} 1; g_i \rightarrow 0 \\ 0; g_i \rightarrow \infty \end{cases}.$$  

Finally, we obtain for $\phi$

$$\tilde{\phi} = \phi (L) \left( 1 + \frac{\lambda - \lambda_L}{\lambda_L} H (g_i L) \right).$$  

One possible choice for $H (g_i L)$ is the exponential function

$$H (g_i L) = \exp (- \rho g_i L),$$

where $\rho$ is a constant which may depend on $\phi (L)$ and/or $R_L$ but is independent of $g_i L$. This choice was carefully studied by solving Eq. (28) numerically (see the Appendix). The results indicate that for all practical cases, $\alpha$ can be represented as [by combining Eqs. (35) and (38)]

$$\alpha = \phi (L) + \frac{\phi (L) / \lambda_L + \beta}{g_i L} + \phi (L) \frac{\lambda - \lambda_L}{\lambda_L} \exp (- \rho g_i L),$$

and consequently $\tilde{\phi}$ becomes

$$\tilde{\phi} = \phi (L) \left( 1 + \frac{\lambda - \lambda_L}{\lambda_L} \exp (- \rho g_i L) \right).$$  

We are now in position to calculate $q(t)$ and $\tilde{q} (t)$ because $\bar{n}$ and $n$ are the same as in the case of the low pumping rate limit [see Eqs. (19) and (20)]. Substituting Eqs. (19) and (20) into Eq. (5) yields

$$q = \frac{g_i L \tilde{\phi} + \phi (L) / \lambda_L}{(g_i L + \beta) \tilde{\phi}}.$$

or, defining $\chi$ as

$$\chi = 1 + \frac{\lambda - \lambda_L}{\lambda_L} H (g_i L),$$

and applying Eq. (37), Eq. (41) becomes

$$q = \frac{g_i L + 1/\chi / \lambda_L}{g_i L + \beta}.$$

Since $H (g_i L)$ is assumed to be a monotonic decreasing function as $g_i L$ increases, we have for $\chi$

$$\frac{\lambda}{\lambda_L} < \chi (g_i L) \leq 1,$$

and consequently $q (g_i L)$ is in the range

$$\frac{\lambda}{\lambda_L} < q (g_i L) \leq 1.$$

This range of $q$ is usually very narrow because $(\lambda / \lambda_L)^{-1}$ is close to 1 unless $R_L < 0.05$. Thus, the fact that the system undergoes losses along the cavity has hardly any effect on the correlation function $q (t)$. Moreover, since the larger $\lambda / \lambda_L$, the larger the saturation effect [1, 2] the fact that $g_i L > 0$ weakens the saturation effect since it shifts $q$ from $(\lambda / \lambda_L)^{-1}$ towards unity.

A larger influence of the losses is expected when considering $\tilde{q}$, the normalized output function. Combining Eqs. (11) and (21) leads to

$$\tilde{q} = \frac{\phi (L)}{\lambda_L} (\beta \tilde{\phi})^{-1},$$

FIG. 1. The boundaries of the normalized output function $\tilde{q}$ with respect to $g_i L$ as a function of mirror reflectances $R_L$ ($R_0 = 1.0$). The dashed line is the upper bound $(\lambda / \lambda_L)^{-1}$. The solid line is the lower bound $(\lambda / \lambda_L)^{-1}$.  

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TABLE I. Results obtained by solving the space-time equations for high pumping rate conditions \( G_0 = 4.8 \times 10^{28} \text{ s}^{-1} (R_0 = 1.0) \).

<table>
<thead>
<tr>
<th>( g_L )</th>
<th>( q_1^* )</th>
<th>( q_2^* )</th>
<th>( q )</th>
<th>( \dot{q} )</th>
<th>( H )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q )</th>
<th>( \dot{q} )</th>
<th>( H )</th>
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<td>0</td>
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<td>0.780</td>
<td>0.780</td>
<td>1.00</td>
<td>0.711</td>
<td>0.970</td>
<td>0.970</td>
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<td>1.00</td>
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<td>1.00</td>
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<td>1.00</td>
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<tr>
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<td>0.797</td>
<td>0.784</td>
<td>0.773</td>
<td>0.99</td>
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<tr>
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<td>0.989</td>
<td>0.969</td>
<td>0.914</td>
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<td>0.988</td>
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<td>0.987</td>
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<td>0.999</td>
<td>0.999</td>
<td>0.723</td>
<td>0.062</td>
</tr>
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</table>

\(^*q_1^*\) are results obtained for \( q \) from Eq. (43) for large \( g_L \), i.e., \( \chi = \lambda / \lambda_L \).

\(^*q_2^*\) are the results obtained for \( q \) from Eq. (43) for small \( g_L \), i.e., \( \chi = 1 \).

or substituting Eqs. (37) and (42) for \( \ddot{q} \) yields

\[ \ddot{q} = (\lambda_L \beta) \gamma^{-1}. \]

Again considering the two limiting cases, \( g_L \sim 0 \) and \( g_L \sim \infty \), we obtain

\[ (\lambda_L \beta)^{-1} < \ddot{q}(g_L, L) < (\lambda \beta)^{-1}, \]

\[ (\lambda \beta)^{-1} \text{ and } (\lambda_L \beta)^{-1}, \]

as a function of \( R_1 \) (for \( R_0 = 1 \)), are presented in Fig. 1. It can be seen that the deviation between the two curves increases as \( R_L \) decreases.

Finally, it should be noted that combining Eqs. (43) and (47) leads to a simple relation between \( q \) and \( \ddot{q} \):

\[ q = \frac{g_L \ddot{q} \beta}{g_L + \beta}. \]

IV. THE QUASI-STeady-STATE MODEL: NUMERICAL VERIFICATION

The aim of the numerical treatment is to solve Eqs. (1) and (2) and extract the three functions \( H(g, L, t) \), \( \dot{q}(t) \), and \( \ddot{q}(t) \) from the results. The function \( H(g, L, t) \) was calculated from Eq. (37) and the functions \( q(t) \) and \( \dot{q}(t) \) from Eqs. (5) and (21), respectively.

Equations (1) and (2) were solved assuming pumping to the upper lasing level only \( (G_1 = 0) \):

\[ G_2 = c \exp(-t/\tau), \]

where \( \tau \) was taken to be 10.0 ns. Two pumping conditions were studied: a high pumping rate \( G_0 = 4.8 \times 10^{28} \text{ s}^{-1} \), and an intermediate pumping rate \( G_0 = 4.8 \times 10^{26} \text{ s}^{-1} \). These values are based on a previous study in which we established that the range of interest for \( G_0 \) is \( 10^{25} < G_0 < 5 \times 10^{28} \text{ s}^{-1} \). This is so because for \( G_0 \) values lower than \( 10^{25} \text{ s}^{-1} \) no threshold could be achieved and for higher values than \( 5 \times 10^{28} \text{ s}^{-1} \) the shape of the output pulse became identical to the shape of the pumping function.

The spontaneous emission term \( S(x, t) \) takes the form

\[ S(x, t) = A \bar{n}_2(x, t), \]

where \( A = 1.0 \times 10^{-9} \text{ cm}^{-1} \). Several other values for \( A \) were tried over the range \( 1.0 \times 10^{-5} < A < 1.0 \times 10^{-15} \) and no significant effect was noticed, except for a small shift of the threshold time which could be neglected. The influence of the spontaneous emission on the threshold was studied before and it was found to be minor for a similar range of \( A \) values.

The results are plotted in Figs. 2–4. The values obtained for the high pumping rate are given in Table I together with those obtained for model calculations. The analysis of the results and the comparison with the QSSM predictions will now be made separately for each of the three functions.

A. Correlation function \( q(t) \)

The behavior of the correlation function \( q(t) \) as a function of \( (\text{dimensionless}) \) time \( t/\tau_m \) (where \( \tau_m \) is the time in which the laser pulse is maximal) is shown in Fig. 2. The straight line was obtained for a high pumping rate, and an oscillating curve was obtained for an intermediate pumping rate. The results were calculated for three absorption loss...
coefficients: no losses $g_iL = 0$, intermediate $g_iL = 1.0$, and large $g_iL = 5.0$. Two mirror reflectivity conditions were considered for each case $R_1 = 0.01$ and $R_L = 0.1$ ($R_0$ was always kept as unity).

Inspection of the results obtained for a pumping rate of $G_0 = 4.8 \times 10^{28} \text{ s}^{-3}$ (high pumping rate) clearly reveals that in all cases the correlation function is time independent during the laser pulse, which means that the system is well within the quasi-steady-state region. The function $q(t)$ obtained for $G_0 = 4.8 \times 10^{28} \text{ s}^{-3}$ (intermediate pumping rate) oscillates with time, which indicates the existence of nonsteady state conditions for this case. In all cases the amplitude of the oscillations is large in the beginning of the laser pulse but then becomes much smaller for larger $t/\tau_m$ values, so that $q(t)$ becomes equal to its QSS value. As $g_iL$ increases the amplitudes at the beginning of the pulse become smaller, but the decay rate towards the steady state value is slower. The same phenomena are observed when the reflectivity is increased, but the behavior is now more relaxed indicating that the system approaches quasi-steady-state conditions more rapidly.

Table I shows the high pumping rate results obtained for $q$ by solving Eq. (5) for several values of $(g_iL)$ for two mirror reflectances $R_L = 0.01$ and $R_L = 0.1$. It is seen that $q_i < q < q_a$, where $q_i$ and $q_a$ are both derived from Eq. (43), the first by substituting $\chi = 1$ (the value of $\chi$ when $g_iL = 0$) and the second by substituting $\chi = \lambda/\lambda_i$ (the value of $\chi$ when $g_iL$ is large). Moreover, the comparison of the numerical results for $q$ with $q_i$ and $q_a$ reveals that for small $g_iL$ values the exact $q$ values are close to $q_a$, but as $g_iL$ increases the $q_i$ values become more reliable.

B. Normalized output function $q(t)$

Figure 3 shows the behavior of the normalized output function $\tilde{q}$ as a function of $t/\tau_m$ ($\tau_m$ is the time when $\phi$ becomes maximal). Here also the straight lines represent the behavior of $\tilde{q}$ in the high pumping rate case $G_0 = 4.8 \times 10^{28} \text{ s}^{-3}$ and the oscillating curves represent the intermediate pumping rate case. The result shown here are also for three values of $g_iL$ and for the two reflectivities.

Consideration of the results for the high pumping rate case indicates that they are time independent like those obtained for $q(t)$. The behavior of $\tilde{q}$ under intermediate pumping rate conditions is quite different. An oscillatory time dependence is noticed with larger amplitudes than those obtained for $q(t)$ calculated for the same parameters. These findings support the conclusion that the system is in nonquasi steady state and, therefore, the analytical treatment made in the previous section cannot be applied here. Nevertheless, a relaxation of the oscillations towards the steady-state value obtained for a high pumping rate is observed in all the cases considered. The rate of relaxation is faster when the losses become smaller, similar to the behavior obtained for $q(t)$. The increase in $R_L$ causes $\tilde{q}$ to be less time dependent.

The (constant) values obtained for $\tilde{q}$ in the high pumping rate case [by solving Eqs. (1) and (2) numerically] are given in Table I. A decrease in $\tilde{q}$ as the losses are increased is seen. Analytically $\tilde{q}$ should vary between $(\lambda/\beta)^{-1}$ (for $g_iL = 0$) and $(\lambda/\beta_L)^{-1}$ (for $g_iL \to \infty$) as given in Eq. (48). This prediction is confirmed by the numerical calculations as the result obtained for $\tilde{q}$ when $R_L = 0.01$ and $g_iL = 0$ is 0.78.
[which is equal to \((\lambda J)^{-1}\), and the result when \(g_i L = 20.0\) is 0.448 which is very close to \((\lambda J)^{-1}\) = 0.426]. This behavior is also obtained for \(R_s = 0.1\) as well as for other \(R_s\) values.

Finally, we would like to comment again on the relation between \(\varphi\) and \(\varphi\tilde{q}\) in the high pumping rate case in Eq. (49). Considering the corresponding values of \(\varphi\) and \(\varphi\tilde{q}\) given in Table I (both, as mentioned earlier, were obtained independently by solving the time-space Eqs. (1) and (2)), one can easily find that the predicted relation between them as given by Eq. (49) is well satisfied.

C. The function \(H(g_s, L)\)

The variation of \(H(g_s, L)\) as a function of \(t/\tau_m\) is plotted in Fig. 4. The results obtained in the high pumping rate case are time independent for all the loss values and reflectances considered. These findings again confirm that the system is in a quasi-steady state. Therefore, the analytical results can be compared with those obtained here.

Inspection of the oscillatory results obtained for an intermediate pumping rate reveals that \(H(g_s, L)\) reflects the nonquasi-steady-state conditions as well. A similar behavior pattern as for \(\varphi\) and \(\varphi\tilde{q}\) is observed, i.e., a faster oscillation relaxation rate when \(g_i L\) is decreased and a lesser time dependence when mirror reflectances are increased. Table I presents the \(H(g_s, L)\) values obtained from exact numerical solution of Eqs. (1) and (2). Analysis of the results reveals that for all cases the \(H(g_s, L)\) values are between the boundaries predicted by Eq. (36) beginning from unity for \(g_i L = 0\) and approaching zero when \(g_i L\) becomes sufficiently large.

The exact \(H(g_s, L)\) values for the high pumping rate case are plotted in Fig. 5 as a function of \(g_i L\) for three reflectivity cases: \(R_s = 0.01, R_s = 0.1,\) and \(R_s = 0.5\) (with \(R_o\) always equal to unity). Figure 5 also shows (the straight lines), for the same reflectivities, the plots of Eq. (35) solved for \(H(g_s, L)\) after substitution of \(\alpha\) obtained by numerical solution of Eq. (28). This procedure was repeated for different radiation intensities \(\phi(L)\) and for each one, \(H(g_s, L)\) was derived as a function of \(g_i L\) (see Appendix). The best agreement with the exact \(H(g_s, L)\) results was obtained for \(\phi(L) = 50.0\), and the corresponding lines are plotted in the figure. The excellent agreement between the exact values and the results obtained from the analytical treatment readily verify the adequacy of the analytical analysis in this case. The exponential representation of \(H(g_s, L)\) [see Eq. (38)] is also well confirmed for practical loss conditions. It is interesting to note that the system is well above saturation for this pumping rate in all the reflectivity cases considered.

V. TREATMENT OF THE OPTICAL EXTRACTION EFFICIENCY

The relevance and the importance of the previous theoretical treatment will now be shown by treating the optical extraction efficiency. We will derive the QSS expressions for the output efficiency and then examine them using results obtained by solving Eqs. (1) and (2). As a final check, comparison will be made with results obtained by other treatments.

A. Derivation of the efficiency formula

The output efficiency at time \(T\) is defined as

\[
\eta = \frac{J_{out}(T)}{J_{in}(T)},
\]

where

\[
J_{out}(T) = \int_0^T \phi(t) dt,
\]

\[
J_{in}(T) = \int_0^T G_d(t) dt,
\]

where \(G_d(t)\) and \(\phi(t)\) were introduced earlier in Eqs. (4) and (6), respectively. Both \(J_{out}\) and \(J_{in}\) are given in units of cm\(^{-1}\). For eximer lasers where the lower lasing level does not exist, the coefficient \(\frac{1}{2}\) that appears in Eq. (52) should be deleted.

A closed form expression for \(\eta\) in this system was obtained from energy conservation considerations. Thus,

\[
\frac{1}{2} J_{in}(T) = J_{out}(T) + \frac{c g_i}{L} \int_0^T dt \int_0^L dx \phi(x, t) dx + \frac{1}{L} \int_0^T dt \int_0^L n_\beta(x, t) dx + \frac{1}{2L} \int_0^T n(x, T) dx + \frac{1}{L} \int_0^T \phi(x, T) dx,
\]

where the total energy pumped into the system is given on the left-hand side and the total energy leaving the system is given on the right-hand side (the first three terms) together with the energy still stored in the cavity (the last two terms). The first two terms stand for the energy that left the cavity as photons, either via the output mirror or as losses. The third term stands for losses due to collisional deactivation processes, the fourth term yields the energy stored as inverted
populations and the fifth as photons.

Equation (54) can be rewritten as

$$\frac{1}{2} J_{in}(T) = J_{out}(T) + e_g \int_0^T \phi(t) dt + \frac{1}{2} \int_0^T \tilde{\phi}_s(t) dt \left( \tilde{n}(T) + \tilde{\phi}(T) \right).$$

Equation (55) is general and applies at every instant for both a pulsed and cw-lasing system. However, in the case of cw lasers, the last two terms can be ignored, whereas for pulsed lasers, if $T$ is large enough, the last term can be deleted but the fourth term may or may not be ignored depending on the circumstances. (In what follows we ignore the fifth term.)

Substituting Eq. (55) in Eq. (52), employing Eq. (21) for the relation between $Q(t)$ and $\phi(t)$ and defining

$$\tilde{n}_s(T) = \frac{1}{T} \int_0^T \tilde{n}_s(t) dt,$$

one obtains

$$\eta = \left( 1 + \frac{g_i L}{\tilde{q}} + \frac{\tilde{n}(T) + 2(T/\tau) \tilde{n}_s(T)}{2J_{out}(T)} \right)^{-1},$$

where $\tilde{q}$ was assumed to be a time independent constant. Equation (57) becomes more attractive if we assume that on the average the lasing process takes place at the threshold and that $\tilde{n}_s = \tilde{n}_e$. Thus [see Eq. (19)],

$$\eta = \left( 1 + \frac{g_i L}{\tilde{q}} + \frac{(1 + 2T/\tau) g_i L + \beta}{2J_{out}(T)} \right)^{-1}.$$  

In cases where $\tau > T$ and $J_{out} > g_i + \beta / L$, Eq. (58) simplifies significantly

$$\eta = \left( 1 + \frac{g_i L}{\tilde{q}} \right)^{-1}.$$  

B. Numerical verification of $\eta$

To examine the validity of the various expressions for $\eta$ we solved Eqs. (1) and (2) to obtain the output energy. Consequently, $\eta$ was calculated using Eq. (52). The exact values of $\eta$ are given as a function of $g_i L$ for several values of $G_o$ and $R_e$ in Figs. 6–9, and as a function of $G_o$ for several values of $g_i L$ and $R_e$ in Figs. 10 and 11. In each of the figures, four different curves obtained by applying either Eq. (57) (assuming $\tau > T$) or Eq. (59), are also given. Most of the results were derived by applying Eq. (59). Only those for low pumping rates, i.e., $G_o < 10^{20}$ s$^{-1}$, were obtained by using Eq. (57) where the values of $\tilde{n}(T)$ and $J_{out}(T)$ were taken from the exact solution of Eqs. (1) and (2).

The four curves differ from each other depending upon the value taken for $\tilde{q}$. The dashed line was obtained by assuming the usual value for $\tilde{q}$ namely $\tilde{q} = 1$. The two solid lines were obtained for $\tilde{q}$ assuming the value of boundaries given by Eq. (48); the upper line was derived for $\tilde{q} = (\lambda \beta)^{-1}$ and the lower for $\tilde{q} = (\lambda_e \beta)^{-1}$. The fourth (dot-dashed) line was obtained by taking the steady state $\tilde{q}$ value as calculated from the numerical solution of Eqs. (1) and (2). These values are listed in Table I.

The following features are to be noted:

1. The efficiency values of all the curves that were calculated with $\tilde{q} = 1$ are much too high. This is particularly true when $R_e = 0.01$.

2. The exact efficiency values $\eta_{ex}$, as a rule, are all found to lie between the two curves obtained with $\tilde{q} = (\lambda \beta)^{-1}$ and $\tilde{q} = (\lambda_e \beta)^{-1}$. Moreover, we also notice [see Figs. 7–9 that $\eta_{ex}$ is very close to $\tilde{q} = (\lambda \beta)^{-1}$] for small values of $g_i L$, but asymptotically approaches $\tilde{q} = (\lambda_e \beta)^{-1}$ for large...
ger values of $g_all$.

(3) The best fit between the exact results and those derived with the simplified formulae is obtained by taking the steady state value of $\bar{q}$. The theoretical curves are seen to go through all the given numerically calculated points. This means that although the lasing process is strongly time dependent, a constant value of $\bar{q}$ is still sufficient to yield the correct overall output efficiency. In particular, it should be emphasized that the corresponding dot-dashed curves in Figs. 10 and 11 were calculated with a single value of $\bar{q}$ for each, namely, Fig. 10(a) with $\bar{q} = 0.748$, Fig. 10(b) with $\bar{q} = 0.958$, Fig. 11(a) with $\bar{q} = 0.530$, and Fig. 11(b) with $\bar{q} = 0.838$ (see Table I).

To summarize these findings we may say that the theory presented in Sec. III was found to yield simple and reliable expressions for the efficiency expression in the case of time dependent lasing systems. The expression simplifies significantly once the pumping rate is high enough or the lasing process long enough.

**FIG. 10.** $\eta$ as a function of pumping rate conditions $G_0$ for low absorption loss $g_l L = 0.5$. Mirror reflectivities (a) $R_0 = 1.0$, $R_L = 0.01$; (b) $R_0 = 1.0$, $R_L = 0.1$. The lines and symbols are the same as in Fig. 6.

**FIG. 11.** Same as Fig. 10, except for high absorption loss $g_l L = 5.0$.

**C. Comparison with experiment and other treatments**

The main motivation for developing the theory presented in previous sections is the need for simplified and reliable expressions to handle pulsed lasing systems. However, there are hardly any experimental or theoretical results available with which to compare our results. Nevertheless, since our theory would apply to cw lasers as well, we will relate to them and discuss two examples.

1. **Comparison with experiment**

Recently Rokni et al.\textsuperscript{15} presented experimental results and a detailed analysis of the KrF laser. They not only supplied all the information they obtained from their experiments but also calculated, among other physical properties, the optical extraction efficiency of their lasing system. In order to calculate the efficiency from our theory, we apply Eq. (58) (neglecting the 1 in the last term as $T \gg \tau$). Thus,

$$\eta = \frac{1 + g_l L}{\bar{q} \beta + \frac{T/\tau (g_l L + \beta)}{L J_{out}(T)}}. \quad (60)$$

To calculate $\eta$, the following numerical values were

**FIG. 12.** $\alpha - \alpha_0/\alpha_1$ as a function of the absorption losses $g_l L$ for different mirror reflectivities and irradiation intensities calculated using the roots of Eq. (27)—see Appendix. Triangles $\phi (L) = 1.0$, circles $\phi (L) = 10.0$. $R_0$ is always 1.0 and (a) $R_L = 0.05$, (b) $R_L = 0.20$, (c) $R_L = 0.50$, and (d) $R_L = 0.80$. 

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used (all are either found in Ref. 15 or can be calculated using the data given therein): \( T = 600 \text{ ns}; \tau = 4.2 \text{ ns}, g_L = 0.35, \) 
\( R_L = 0.79; L = 100 \text{ cm}, \sigma = (2.5 \pm 0.3) \times 10^{-16} \text{ cm}^2, \) 
\( E_{\text{out}} = 102 \text{ J}, h\nu = 4.98 \text{ eV}. \) The number of photons which left the cavity through the output mirror is found to be 
\( \rho_{\text{out}} = 1.5 \times 10^{16} \text{ cm}^{-1} \) and consequently 
\[ J_{\text{out}} = \alpha \rho_{\text{out}} = 3.75 \pm 0.45 \text{ cm}^{-1}. \] (61)

Using the value of \( R_L = 0.79, \) we obtain \( \beta = 0.78. \) \( \tilde{q} \) is assumed to be equal to 0.98 (based on the results presented in Table 1). Substitution of all numerical values in Eq. (60) leads to \( \eta = 0.53 \pm 0.02, \) which is reasonably close to the values of Rokni et al. \(^{15}\) (\( \eta = 0.46), \) Schindler \(^3\) (\( \eta = 0.48), \) and Rigrod \(^3\) (0.48).

2. Comparison with other theoretical treatments

Schindler\(^3\) also calculated \( \eta \) for a scaled up cavity, i.e., \( g_L L = 1.4, \) and \( R_L = 0.01. \) Assuming that the total output increased by the same factor as the photon density, we obtain 
\[ J_{\text{out}} = 12.8 \text{ cm}^{-1} \] (versus 3.75 cm\(^{-1}\) in the previous case).

From Table 1 one finds by interpolation that the value of \( \eta \) for \( g_L L = 1.4 \) and \( R_L = 0.01 \) is 0.69. Substituting all the relevant numerical values in Eq. (60) yields \( \eta = 0.44. \) This value is larger than Schindler's result (0.41) by 7%. Although the discrepancy between Schindler's results and ours is not large, the fact that it is consistent is still somewhat disturbing. However, since our theory was carefully tested against numerical exact results, we are led to believe that our values are relevant and that Schindler's results are somewhat too low.

VI. SUMMARY AND CONCLUSIONS

In the present work the quasi-steady-state model\(^{1,2}\) was extended to include absorption losses along the laser cavity. It was found analytically that in the small signal gain region the correlation function and the normalized output function are not affected by the presence of losses and therefore remain equal to unity.

For the high pumping region, quite different behavior was observed. The analytical treatment reveals for this region that the losses have an opposite effect on the correlation function \( g(t) \) and the normalized output function \( g(L). \) The absorption losses tend to lower the space variation of the radiation and consequently the deviation of the correlation function from unity is reduced. Thus, increasing the losses has the same effect on \( g(t) \) as increasing the mirror reflectivities. On the other hand, the absorption losses reduce the useful part of the radiation and consequently the deviation of \( \tilde{q} \) from unity becomes larger as \( g_L L \) increases.

This finding can be of importance in practical cases. In the ordinary numerical simulation, \( \tilde{q} \) [see Eq. (21)] is always taken to be unity. Our analytical (and numerical) treatment indicates that this is not correct, i.e., \( \tilde{q} \) is always smaller than 1 and the deviations become larger than 0.1 when \( R_L < 0.1 \) and \( g_L L > 2. \)

One of the main achievements of the analytical part of this work is the explicit representation of \( \tilde{g} \) in the case where absorption losses are included in the steady-state equations.

This was accomplished largely by introducing a new function \( H(g_L L), \) which for all practical cases, was found to be an exponential function.

All the analytical results were verified by comparison with the results obtained from numerical solution of the exact space-time equations for the three-level system under high and intermediate pumping rate conditions.

We also compared our theory with other treatments. This was done through the optical extraction efficiency term \( \eta. \) Using our approach we constructed an analytical expression for \( \eta \) which was then carefully examined employing results obtained by integrating the time-space equations [Eqs. (1) and (2)]. Using the same formula we calculated the optical extraction efficiency for the KrF laser for two cases. In both of them the discrepancy between our result and those obtained from other treatments was at most 10%.

APPENDIX: THE EXPLICIT REPRESENTATION OF \( \alpha \)

In the text we arrived at an explicit representation for \( \alpha, \)
\[ \alpha = \phi(L) + \frac{\phi(L) / \lambda_L + \beta}{g_L L} + \frac{\lambda - \lambda_L}{\lambda_L} H(g_L L), \] (A1)
where \( H(g_L L) \) fulfills the following requirements (but otherwise is unknown):
\[ \lim_{g_L L \to 0} H(g_L L) = \begin{cases} 1; & g_L L \to 0 \\ 0; & g_L L \to \infty \end{cases} \] (A2)

To examine the validity of Eq. (A1) we define a new parameter \( z = p / \alpha. \) Substituting \( F(z) \) from Eq. (27) into Eq. (28), and expressing the resulting equation in terms of \( z \) (always taking \( R_L \) to be unity), we obtain
\[ G(z) = \ln \left[ \frac{b - z + [1 - z^2][b - 1]^{1/2}}{1 - b z} \right] = 1 + \frac{z^2}{\rho} \] (A3)
where \( 0 < z < 1/b \) and \( b = (1 + R_L) / 2(R_L) \) \(^{1/2}. \) Equation (A3) was solved numerically for \( z \) to determine \( \alpha \) for different values of \( R_L, \phi(L) \) and \( g_L L. \) We define \( \alpha_0 \) and \( \alpha_1 \) as
\[ \alpha_0 = \phi(L) + \frac{\phi(L) / \lambda_L + \beta}{g_L L}, \] (A4)
\[ \alpha_1 = \frac{\lambda - \lambda_L}{\lambda_L}, \]

such that substitution into Eq. (A1) results in
\[ \frac{\alpha - \alpha_0}{\alpha_1} = \frac{H(g_L L)}{H(g_L L)}, \] (A5)
\( (\alpha - \alpha_0) / \alpha_1 \) is plotted semilogarithmically in Fig. 12 as a function of \( g_L L \) in the range \( 0 \leq g_L L \leq 10 \) for different values of \( R_L \) and \( \phi(L). \) The most striking result is that, for all cases of interest, \( H(g_L L) \) is well approximated by an exponential function:
\[ H(g_L L) = \exp(-\rho g_L L), \] (A6)
where \( \rho \) (the slope of the lines in Fig.12) is dependent on \( \phi(L) \) and \( R_L \) but is independent of \( g_L L. \) From Fig. 12 it can be seen that \( \rho \) increases as either \( \phi(L) \) increases or \( R_L \) decreases.